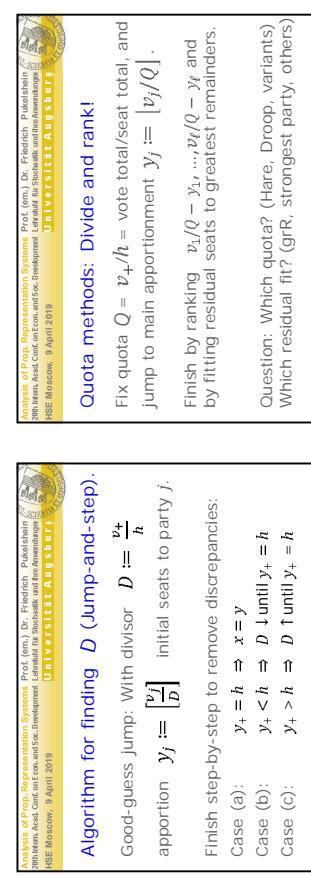
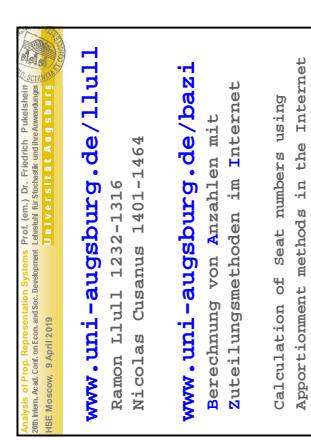
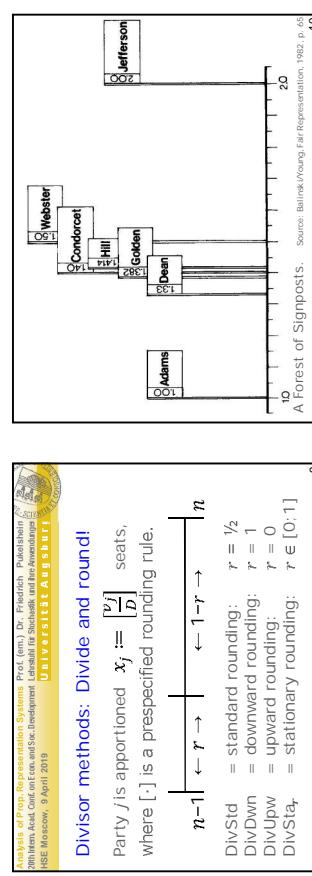
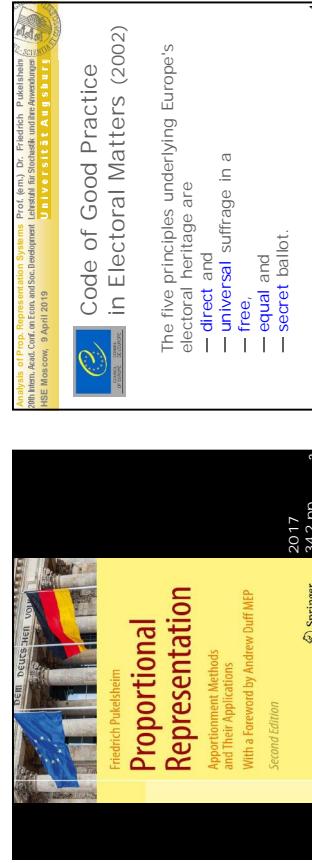
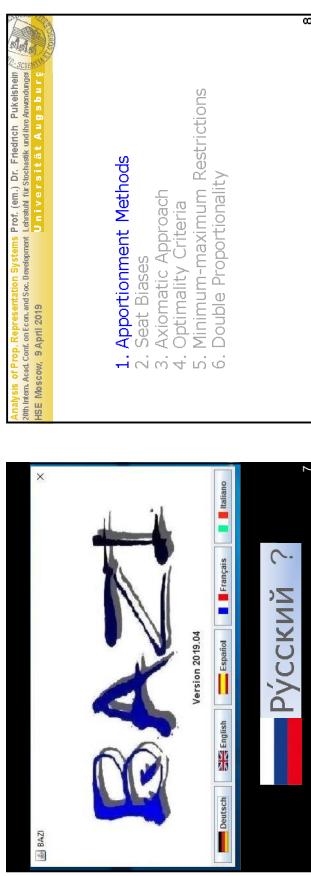
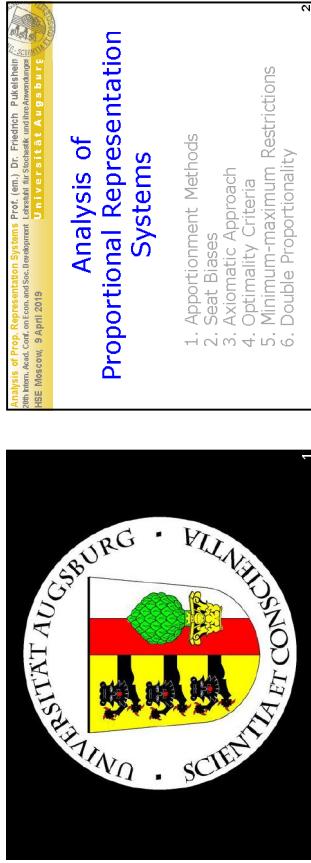


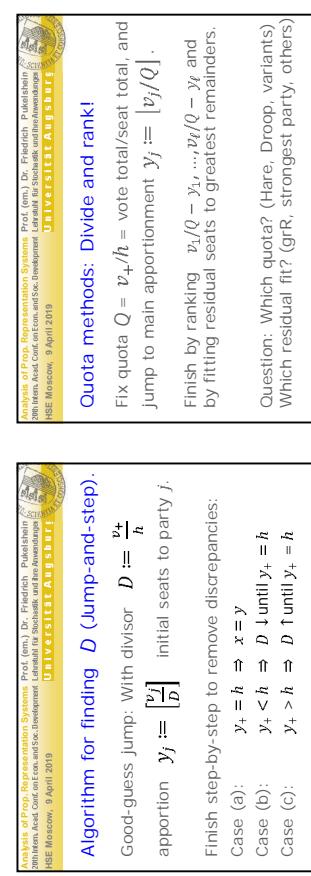
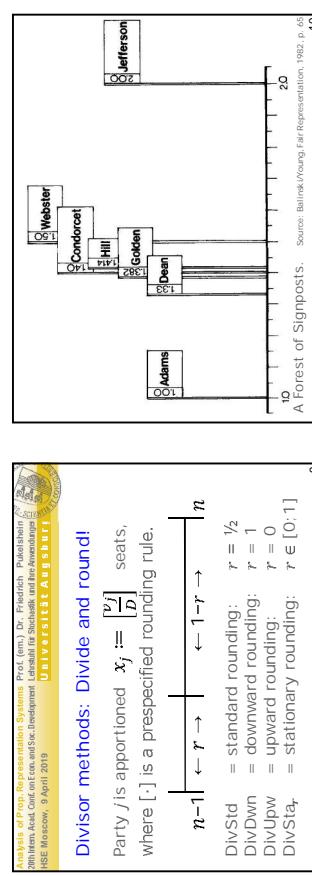
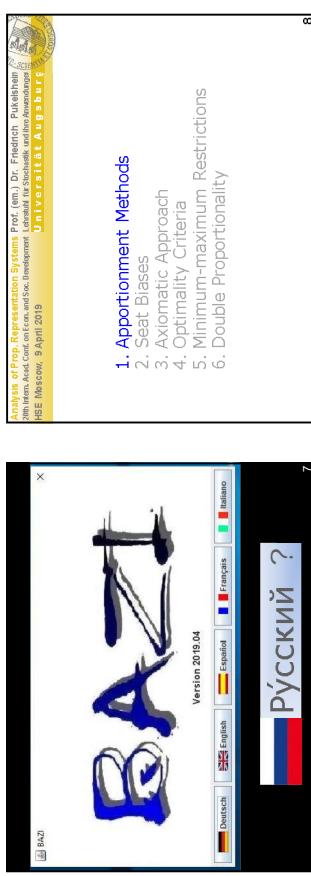
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Analysis of PR Systems



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Divisor methods:
Fixed rounding rule,
flexible electoral key ("divisor")
– **recommendable**

Quota methods:
Fixed electoral key ("quota"),
flexible rounding rule
– **prone to paradoxes**

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Divisor methods:
1. Apportionment Methods
2. **Seat Biases**
3. Axiomatic Approach
4. Optimality Criteria
5. Minimum-maximum Restrictions
6. Double Proportionality

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The seat bias of the k -strongest party is

$$B(k) := \lim_{h \rightarrow \infty} E[\alpha_k - w_k h] | w_1 \geq \dots \geq w_\ell \geq 1].$$

Theorem (Three-factor formula).
Div/Sta_r, with split $r \in [0; 1]$, has

$$B(k) = \left(r - \frac{1}{2}\right) \frac{1}{k} + \dots + \frac{1}{k} - 1 \left(1 - \ell t\right).$$

Proof. Schuster/Puk./Drton/Draper (2003), Pólya (1918)

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George Pólya
1887-1985



A geometric diagram showing a triangle divided into smaller triangles by internal lines. Some of these smaller triangles are shaded in yellow, red, and blue, illustrating a voting system or apportionment method.

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Theorem (Invariance principle).
If the vote share distribution is absolutely continuous, then the rounding residuals satisfy

$$(u_{r,1}(h), \dots, u_r(h), w_1, \dots, w_\ell) \xrightarrow{h \rightarrow \infty} (u_1, \dots, u_\ell, w_1, \dots, w_\ell)$$

Proof. Heinrich/Puk./Schwingenschild (2005),
Janson 2013, Tukey 1938 (Math. USSR Sbornik 4).

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Global χ^2 -Optimality Theorem (DivStd).
 $f(x) = \min_{y \in \mathcal{Y}} f(y) \Leftrightarrow x = \text{DivStd}(h; w_1, \dots, w_\ell)$

With vote weights $w_j := v_j/v_+$, consider the χ^2 -type goodness-of-fit criterion

Proof. Sainte-Laguë 1910.

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Consider the pairwise disparity criterion

$$g(x_j, x_k) := \left| \frac{x_j/h}{w_j} - \frac{x_k/h}{w_k} \right|$$

Pairwise Disparity Theorem (DivStd).

$$g(x_j, x_k) \leq g(x_j - 1, x_k + 1) \quad \forall j \neq k$$

$$\Leftrightarrow x = \text{DivStd}(h, w_1, \dots, w_\ell)$$

Proof. Bortkiewicz 1919.

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André Sainte-Laguë
1882-1950


Daniel Webster
1782-1852


Ladislaus von Bortkiewicz
1868-1931


Hans Scheipers
* 1928

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Double proportionality

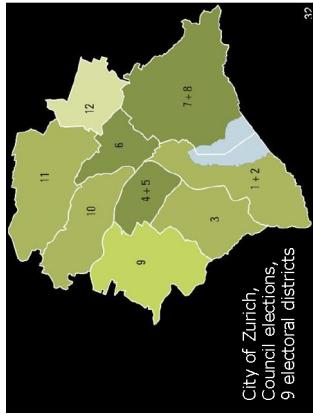
Doubly proportional divisor methods capture two dimensions of the apportionment task (Balinski/Demange 1989):

- vote counts of parties, and
- population figures of districts.

World premiere in Zurich, 12 February 2006



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1. Apportionment Methods

1. Seat Biases
2. Axiomatic Approach
3. Optimality Criteria
4. Minimum-maximum Restrictions
5. Double Proportionality

Seat apportionments that must obey minimum requirements and maximum capping

- are easily obtained using **divisor methods**,
- but awkwardly handled by **quota methods**.

For divisor methods, the underlying rounding rule $\lfloor \cdot \rfloor^b$ is simply modified into the “truncated rounding rule” $\lfloor \frac{\nu_j}{D_j} \rfloor^b := [\text{median}(q_j, \frac{\nu_j}{D_j})]$.

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Theoretically, seat numbers are unrestricted:

$$x_j \in [0, \dots, h_j]$$

Practically, minimum requirements q_j and/or maximum cappings b_j may be essential:

$$x_j \in \{q_j, \dots, b_j\}$$

USA – at least 1 seat for every Union State
CH – at least 2 seats for every Swiss Canton
EU – at least 6 seats for every Member State
EU – at most 96 seats for every Member State

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With $z_{ij} := \left[\frac{v_j}{C_i D_j} \right]$ for $i \leq k$ and $j \leq \ell$, consider

$$g_v(C, D) := \left(\prod_{i \leq k} C_i^{z_{i+1} - r_i} \right) \left(\prod_{j \leq \ell} D_j^{z_{i+j} - s_j} \right) f_v(D)$$

Minimax-Theorem.

$$\min_{x \in \mathbb{N}_{\geq q_{r,s}}^k} f_v(x) = \max_{\substack{C \in (0, \infty)^k \\ D \in (0, \infty)^{\ell}}} g_v(C, D)$$

Proof. Gaffke/Puk. 2008.

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Iterative (dual) algorithms:

- in floating point arithmetic,
- in integer arithmetic

AS	alternating scaling algorithm
TT	tie-and-transfer algorithm
AS TT	hybrid version: AS piped into TT

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Minimize $f_v(x) := \prod_{i \leq k} x_i z_{i+1} / x_i z_{i+1} - 1$

$$\frac{(x_i z_{i+1})^b}{v_{i+1}} \quad \text{for } x \in \mathbb{N}^{k \times \ell}$$

with $x_{i+1} = r_i$

and $x_{i+j} = s_j$

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